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Regge Trajectories for Yukawa Potentials*

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Regge trajectories for attractive and repulsive Yukawa potentials are investigated. Certain analytic relations satisfied by the trajectories are derived. Results obtained by an IBM-7090 computer program are given in the form of figures.

IN a previous paper[1] we presented some preliminary results concerning poles of the scattering amplitude in the complex angular momentum plane. We gave partial descriptions of the trajectories of the poles for an attractive Yukawa potential. In the present paper we extend this work, and we also consider a repulsive potential and comment on the effect of a hard core which dominates the centrifugal barrier at the origin. In the case of potentials that can be represented by a superposition of Yukawa potentials, we derive certain relations between the energy at which a Regge curve can cross the real l axis and the potential strength. These crossing points are shown to be indeterminacy points in the S matrix, which play a vital role in the behavior of the Regge trajectories in the left-hand λ plane (where $\lambda=l+\frac{1}{2}$).

We consider the Schrödinger equation

$$\left[\frac{d^2}{dr^2}+k^2-V(r)-\frac{l(l+1)}{r^2}\right]u_l(r)=0\,, \tag{1}$$

whose solution $u_l(r)$ satisfies the boundary conditions

$$u_l(r) \underset{r\to 0}{=} r^{l+1}\,,$$

$$u_l(r) \underset{r\to\infty}{=} e^{-ikr}-S(l,k)e^{-i\pi l}e^{ikr}. \tag{2}$$

The S matrix can then be written in the form

$$S(l,k) \underset{r\to\infty}{=} \frac{(u'+iku)e^{-ikr}}{(u'-iku)e^{ikr}}e^{i\pi l}. \tag{3}$$

The meromorphy of $S(l,k)$ has been investigated by Boltino *et al.*[2] and Squires,[3] with the potential $V(r)$ subject to the following three conditions:

(i) $\quad V(r)=\int_{m>0}^{\infty} d\mu\,\sigma(\mu)e^{-\mu r}/r\,,$

(ii) $\quad \int_0^{\infty} d\rho\,\rho\,|V(\rho e^{i\theta})|<\infty\,,$ for all $|\theta|<\pi/2\,,$

(iii) $\quad rV(r)$ regular at $r=0\,.$

Under these conditions the S matrix is meromorphic in the full l plane and in the k plane cut along the imaginary k axis.

We consider potentials that can be represented by superpositions of Yukawas:

$$rV(r)=\int_{\mu_0}^{\infty}\sigma(\mu)e^{-\mu r}dr\,,\quad \mu_0>0\,, \tag{4}$$

where we assume

$$\int_{\mu_0}^{\infty}\sigma(\mu)\mu^n d\mu<K \quad \text{for all } n. \tag{5}$$

We can then make the expansion

$$rV(r)-k^2r=\sum_{n=0}^{\infty}\alpha_n r^n. \tag{6}$$

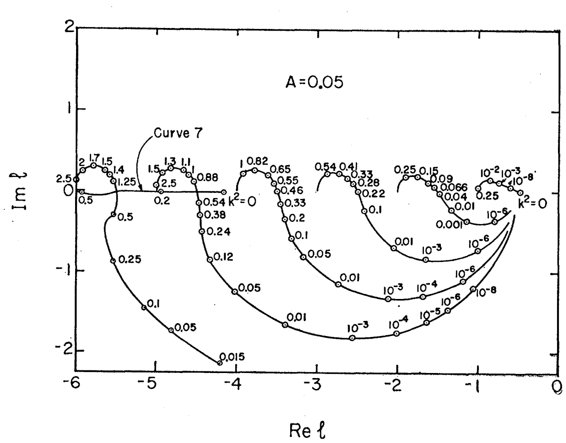

FIG. 1. l-plane plots of the first six Regge trajectories for the potential strength $A=0.05$. Some k^2 values are written adjacent to the curves.

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[1] A. Ahmadzadeh, P. G. Burke, and C. Tate, Lawrence Radiation Laboratory Report UCRL-10140, 1962 (unpublished). See also C. Lovelace and D. Masson, Nuovo Cimento **26**, 472 (1962).

[2] A. Bottino, A. M. Longoni, and T. Regge, Nuovo Cimento **23**, 954 (1962).

[3] E. J. Squires, Nuovo Cimento **25**, 242 (1962).

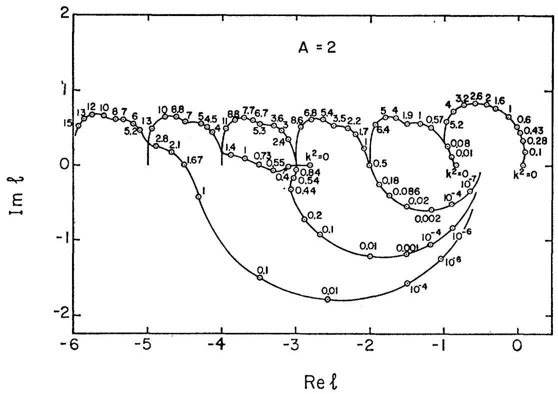

FIG. 2. l-plane plots of the first six Regge trajectories for the potential strength A=2. Some k^2 values are written adjacent to the curves.

The wave function

$$u_l(r)=r^{l+1}\sum_{n=0}^{\infty}a_nr^n$$

and the coefficients a_n are given by the following recurrence relation:

$$a_n=\frac{1}{2l+1+n}\cdot\frac{1}{n}\sum_{m=0}^{n-1}\alpha_m a_{n-1-m}\,,\quad n\geqslant 1\,,\quad a_0=1\,. \tag{8}$$

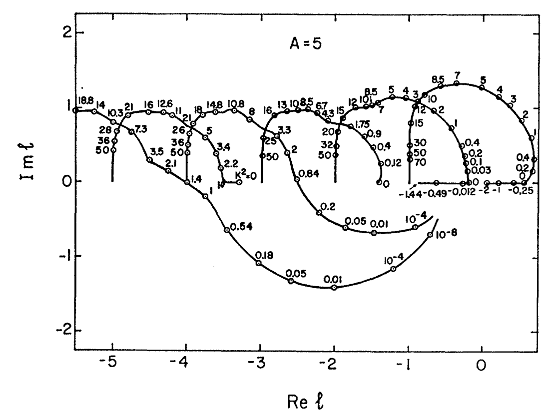

FIG. 3. l-plane plots of the first six Regge trajectories for the potential strengths $A=5$. Some k^2 values are written adjacent to the curves.

According to Eq. (3), a pole of the S matrix occurs when

$$(u'-iku)|_{r_0}=0\,, \tag{9}$$

where r_0 is chosen outside the interaction radius. When both l and k are real, it follows from Eqs. (1) and (2) that u and u' are real. Therefore, (9) cannot vanish unless u vanishes identically for all r. Now we can express the distance Δl of a pole in S from the real l axis, for k^2 real and positive, as

$$\Delta l\approx\frac{(u'-iku)|_{r_0}}{(d/dl)(u'-iku)|_{r_0}}\,, \tag{10}$$

where the numerator and denominator in (10) are evaluated at the point on the real l axis that is nearest to the pole. Provided the denominator does not have a pole and since, in general, we expect u and u' to be continuous functions of l, then Δl cannot become indefinitely small. We expect, therefore, that Regge poles will not be able to migrate across the real l axis as a function of k^2 (real and positive).

An exception to the conditions of continuity on u and u' occurs at certain "indeterminacy points" of the S matrix. Similar points have been noticed by Barut and

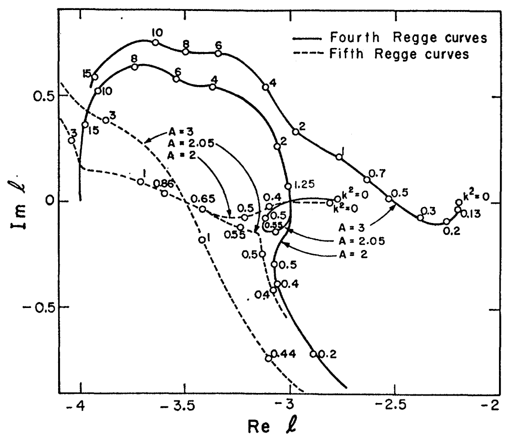

FIG. 4. Parts of Regge curves 4 and 5 for $A=2$, 2.05, and 3, showing the transition for curve 4 of the point at which it crosses the real l axis.

Calogero[4] in their work on a square-well potential, but in that case the points do not appear to play such a vital role in the behavior of the Regge trajectories. We can see how these points arise by considering Eq. (8). A pole in the wave function u occurs when $2l+1+n=0$; i.e., at the points $l_0=-1,\,-\frac{3}{2},\,-2,\,\cdots$. Let us rewrite Eq. (7) for a value of l in the neighborhood of one of these points:

$$u_l(r)=r^{l+1}\left\{\sum_{n=0}^{\infty}b_nr^n+\frac{1}{l-l_0}\sum_{n=-2l_0-1}^{\infty}c_nr^n\right\}. \tag{11}$$

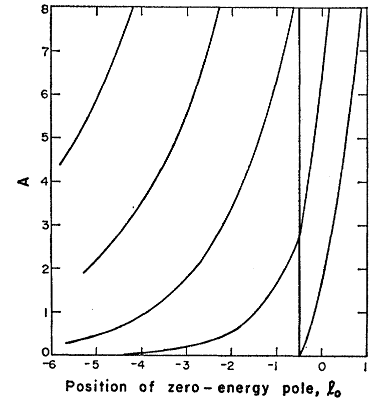

FIG. 5. Positions on the real l axis of the first few zero-energy poles as functions of the potential strength A. The vertical line at Re$l=-\frac{1}{2}$ represents an accumulation point of an infinite number of poles for all A.

[4] A. O. Barut and F. Calogero, Phys. Rev. **128**, 1383 (1962).

We include in the second term of (11) those terms (and only those terms) of (7) which have a pole at $l=l_0$. The rest are included in the first term of (11). We note now that the b_n and the c_n are regular in the neighborhood of $l=l_0$.

The condition for a pole in the S matrix, given by (9), now becomes

$$\left[(u_1'-iku_1)+\frac{1}{l-l_0}(u_2'-iku_2)\right]_{r=r_0}=0, \qquad (12)$$

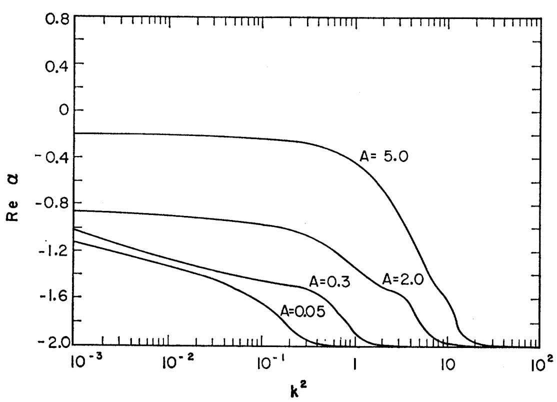

FIG. 6. Reα versus k^2 for curve 2.

where we have written u_1 and u_2 for the first and second terms in (11), respectively. If $(u_2'-iku_2)\propto(l-l_0)$ as l approaches l_0, then (12) can obviously be satisfied. This is a condition on k^2 through expansion (6), and the S matrix can be made to take any value by altering the proportionality constant. In the limit $l=l_0$ our condition obviously reduces to

$$c_{-2l_0-1}=-\frac{1}{2(2l_0-1)}\sum_{m=0}^{-2l_0-2}\alpha_m a_{-2l_0-2-m}=0. \qquad (13)$$

This can be satisfied at the points where there are at least two terms in the summation (13). If α_0 is nonzero, this means that $l_0=-\frac{3}{2}, -2, -\frac{5}{2}, \cdots$ are indeterminacy

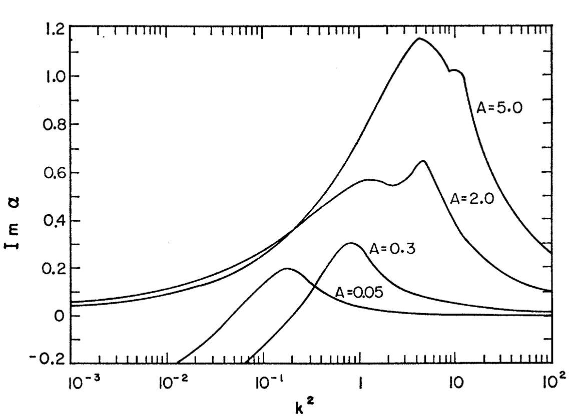

FIG. 7. Imα versus k^2 for curve 2.

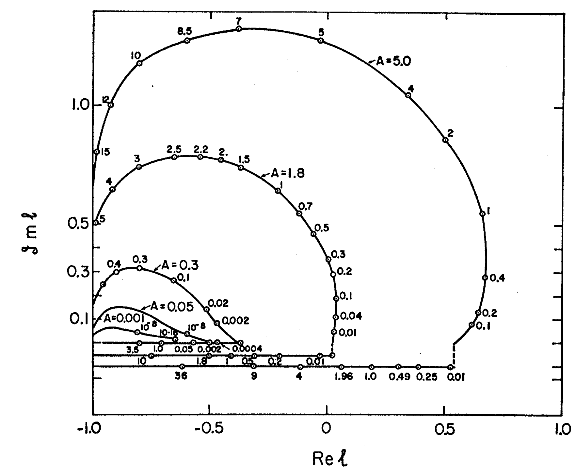

FIG. 8. The first Regge curve for several potential strengths. The k^2 values are written adjacent to the curves. (When k^2 is negative, some of these values are shown displaced from the real axis for the sake of clarity.)

points at the values of k^2 satisfying (13). Writing (13) explicitly at the points $-\frac{3}{2}$, -2, $-\frac{5}{2}$, we obtain

$$\alpha_0^2-\alpha_1=0, \text{ for } l_0=-\tfrac{3}{2};$$
$$\alpha_0^3-4\alpha_0\alpha_1+4\alpha_2=0, \text{ for } l_0=-2; \qquad (14)$$
$$\alpha_0^4-10\alpha_0^2\alpha_1+24\alpha_0\alpha_2+9\alpha_1^2-36\alpha_3=0, \text{ for } l_0=-\tfrac{5}{2}.$$

As an example, let us consider the potential

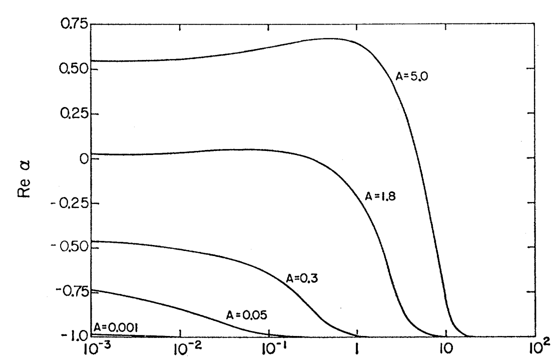

FIG. 9. Reα versus k^2 for curve 1 when k^2 is positive.

$V(r)=-Ae^{-r}/r$. Then these relations become

$$k^2+A^2-A=0, \quad l_0=-\tfrac{3}{2};$$
$$4k^2+A^2-4A+2=0, \quad l_0=-2; \qquad (15)$$
$$9k^4+(10A^2-18A)k^2+A^4-10A^3+21A^2-6A=0, \quad l_0=-\tfrac{5}{2}.$$

These equations are necessary and sufficient conditions that Regge curves should cross the axis at the energies

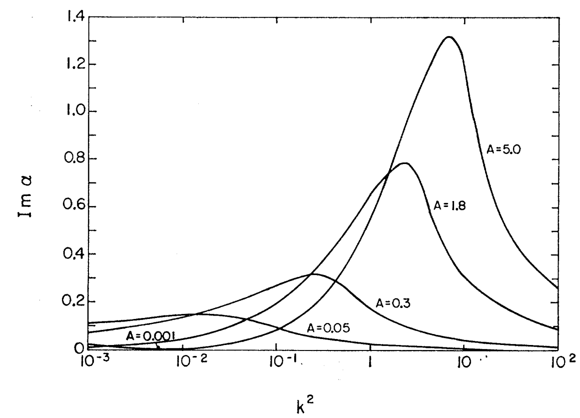

FIG. 10. Imα versus k^2 for curve 1 when k^2 is positive.

given by their solution. The first equation in (15) states that a Regge curve must cross the real l axis at $-\frac{3}{2}$ for k^2 real and positive when A lies between 0 and 1. For $A>1$ the curve must pass through this point at a negative value of k^2; i.e., on its path along the real l axis. Many such statements can be made on the basis of Eqs. (15). In particular it can be shown that for small A, Regge curves cross the axis at a finite value of k^2 (which approaches zero as A goes to zero) at the points $-\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \cdots$ but not at $-2, -4, -6, \cdots$

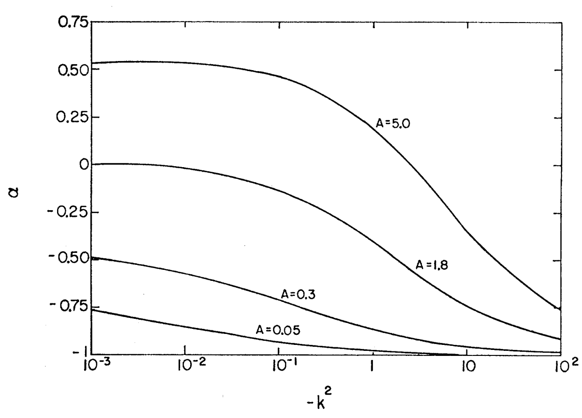

FIG. 11. Reα versus k^2 for curve 1 when k^2 is negative.

(see Fig. 1). Also as $A\rightarrow\infty$, Eqs. (15) give the standard Coulomb behavior $n|k|=A$ for a pole returning to the negative integer $-m$, where the indeterminacy point is given by $l_0=-m+(A/n|k|)$.

As a specific example, we considered the Regge trajectories for a single Yukawa potential $-Ae^{-r}/r$, and we allowed the strength parameter A to assume positive and negative values. Our method, involving the numerical solution of Eq. (1) on the IBM-7090 computer, has been sketched in Ref. 1. We present our results in the figures. In Fig. 1 we give the l-plane plot for $A=0.05$. The important features shown are: (a) In the high-energy limit ($k^2\rightarrow\infty$) the curves approximate to the Coulomb case at the negative integers, (b) the first few curves cross the real l axis at the first possible negative half integer and go to $l=-\frac{1}{2}$ as $k^2\rightarrow 0$, and (c) in the limit $A\rightarrow 0$ it appears that this feature is shared by all curves except the first. The first curve always extends to the right of $l=-\frac{1}{2}$ for $k^2=0$. We, thus, find the point $l=-\frac{1}{2}$ an accumulation point for an infinite number of zero-energy poles for all A positive and negative. Figures 2 and 3 show the l plane for $A=2$ and $A=5$, respectively. It is seen that as A is increased, further zero-energy poles move along the real l axis from $-\infty$

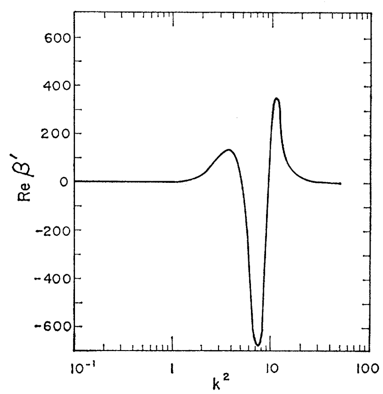

FIG. 12. Re($\beta e^{-i\pi\alpha}$) versus k^2 for curve 1. $A=5$.

and become associated with each Regge curve in turn. The transition from one curve to another is shown for one case in Fig. 4. In Fig. 5 we show the position of some zero-energy poles as a function of A. The vertical line at Re $l=-\frac{1}{2}$ represents an infinite number of poles. As A is increased, the second zero-energy pole becomes associated with the second Regge curve at $l=-\frac{3}{2}$, the third with the third curve at $l=-2$, and the fourth with the fourth curve at $l=-\frac{5}{2}$, etc. These relations can also be seen from Eqs. (15). Once a curve has picked up its ultimate zero-energy pole it can no longer cross the axis and is constrained to move into the right-half l plane with Im l always positive. Both the transition shown in Fig. 4 and the existence of quadratic and higher order equations in k^2 in (15) show that double poles in the S matrix can and do occur. We must, consequently, be

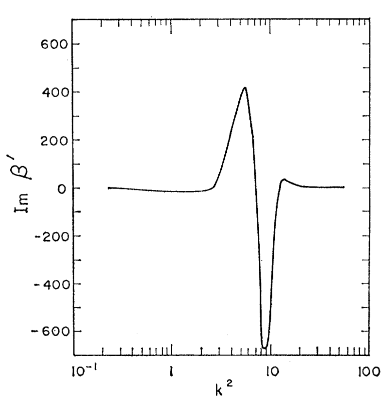

FIG. 13. Im($\beta e^{-i\pi\alpha}$) versus k^2 for curve 1. $A=5$.

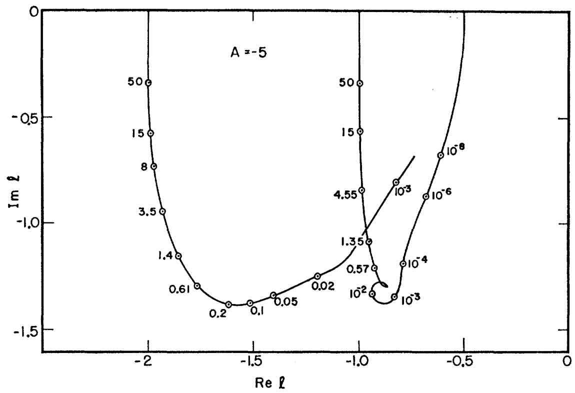

FIG. 14. The first two Regge curves for the repulsive Yukawa potential $A=-5$ and for k^2 positive.

cautious in applying Taylor's[5] proof of the analyticity sf $\alpha(k^2)$ and $\beta(k^2)e^{-i\pi\alpha(k^2)}$ in the k^2 plane. In fact, certain extra branch cuts may appear. Such effects are apparent for all Regge curves except the first, when A becomes oufficiently large and positive (Figs. 6 and 7). Figure 8 shows the first Regge curve for several values of A, and Figs. 9, 10, and 11 give the real and imaginary parts of this trajectory plotted against k^2.

In Figs. 12 and 13 we plot the real and imaginary parts of $\beta(k^2)e^{-i\pi\alpha(k^2)}$ for curve 1. This is a real analytic function with no left-hand cut, and it appears from our results that it satisfies a usual type of dispersion relation. Figure 14 gives the first two Regge curves for the repulsive potential $5e^{-r}/r$. It is interesting to note that both curves go to the point $l=-\frac{1}{2}$ as $k^2\to 0$. Equations (15) show that no curve can pass through the points $l=-\frac{3}{2}, -2, -\frac{5}{2}$ for $A<0$ and $k^2>0$. It may be true that for $A<0$ and $k^2>0$ no curve can cross the axis and that all must go to $l=-\frac{1}{2}$, but we have not obtained a general proof of this.

In concluding this part of the work we remark that if $\alpha_0=0$ in expansion (6) we still get indeterminacy points, but they now satisfy $l_0\leqslant -\frac{5}{2}$. In general, increasing the dominant power of r in the potential for small r pushes to the left the first indeterminacy point of the S matrix in the complex l plane. This seems to fit in with Bethe's[6] result that the high-energy limits of the Regge trajectories are also pushed to the left in this case.

The foregoing analysis shows that Regge trajectories in the left-half λ plane have an exceedingly complicated behavior due to the appearance of certain indeterminacy points. An interesting fact pointed out by Predazzi and Regge[7] is that a repulsive core, whose strength is sufficient to dominate the centrifugal barrier at $r=0$, causes the following symmetry in the S matrix:

$$S(\lambda,k)=S(-\lambda,k)e^{2i\pi\lambda}. \tag{16}$$

In that case the left-half λ plane is no longer interesting. We investigated the effect of adding such a repulsive core to the single Yukawa. We considered

$$V(r)=\frac{e^{-\mu r}}{r^4}-\frac{5e^{-r}}{r},$$

and took both $\mu=0$ and $\mu=1$. In both cases the Regge trajectories appear to be unbounded as $k^2\to\infty$. We followed several trajectories up to an energy of about $k^2=50$ and found that both real and imaginary parts of λ were still increasing.

We regard this as an indication that a hard core of the type (17) is unphysical. It may in fact be necessary to take seriously the investigation of Regge trajectories in the left-half λ plane.

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[5] J. R. Taylor, Phys. Rev. **127**, 2257 (1962).

[6] H. A. Bethe and T. Kinoshita, Phys. Rev. **128**, 1418 (1962).

[7] E. Predazzi and T. Regge, Nuovo Cimento **24**, 518 (1962).